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Sequence characterization of almost-Riordan arrays

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A B S T R A C T

In this study, we consider A, Z and ω-sequences of the almost-Riordan arrays and their inverses. Also, we get A, Z and ω-sequences of the product of two almost-Riordan arrays. Then, by defining the sum of two almost-Riordan arrays, we have A, Z and ω-sequences of this sum.

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1. Introduction

Riordan arrays play an important role in combinatorics in terms of obtaining combinatorial identities. In addition, Riordan arrays have applications in many fields of mathematics. Riordan arrays are infinite lower triangular matrices defined by formal power series.

Let us consider the following formal power series

$$g(x) = g_0 + g_1 x + g_2 x^2 + \dots$$

and

$$f(x) = f_0 + f_1 x + f_2 x^2 + \dots$$

with $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$. The generating function of the kth column of Riordan matrix is defined as follows

$$g(x)(f(x))^k, \quad k = 0, 1, 2, \dots$$

Also, a Riordan matrix is represented as a pair of the formal power series with $D = (g(x), f(x))$. The multiplication of two Riordan matrices is defined by

$$(g(x), f(x))(h(x), l(x)) = (g(x)h(f(x)), l(f(x))). \tag{1.1}$$

The set of Riordan matrices is a group under matrix multiplication with (1.1). This group is called Riordan group and denoted with R.

The identity element of the Riordan group is

$$I = (1, x) \tag{1.2}$$

and the inverse of $(g(x), f(x))$ is

$$(g(x), f(x))^{-1} = \left(\frac{1}{g(\overline{f}(x))}, \overline{f}(x) \right) \tag{1.3}$$

where $\overline{f}(x)$ is the compositional inverse of $f(x)$ in [13].

The characterizations of Riordan arrays are given in [10] and [12]. We can give these characterizations with the following theorem.

Theorem 1.1. *Let $D = (d_{n,k})_{n,k \geq 0}$ be an infinite lower triangular matrix. D is a Riordan matrix if and only if there exist two sequences $A = \{a_0, a_1, a_2, \dots\}$ and $Z = \{z_0, z_1, z_2, \dots\}$ with $a_0 \neq 0$ and $z_0 \neq 0$. These sequences are defined as follows,*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \dots \tag{1.4}$$

and

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \dots \tag{1.5}$$

where the coefficients $\{a_0, a_1, a_2, \dots\}$ *and* $\{z_0, z_1, z_2, \dots\}$ *are known as the* A *and* Z*-sequence of the Riordan matrix* $D = (g(x), f(x))$*, respectively.*

Let $A(x)$ and $Z(x)$ be the generating function of A and Z-sequences of the Riordan matrix $D = (g(x), f(x))$. Then, the following identities hold,

$$f(x) = xA(f(x)) \tag{1.6}$$

and

$$g(x) = \frac{g_0}{1 - xZ(f(x))} \tag{1.7}$$

in [10].

More information on the Riordan arrays and A and Z-sequences of the Riordan arrays can be seen in [4,5,8,10,12,13,15]. Many researchers have studied the Pascal matrices with the Riordan arrays. In [9], the authors considered the generalizations of Pascal matrix, Fibonacci and Pell matrix and obtained factorizations of them by using Riordan arrays. You can find detailed information about Pascal matrices with Riordan arrays in [7,16,17].

Recently, the generalization of the Riordan arrays have been studied. One of them is the almost-Riordan arrays. In [2], Barry states that an ordered triple of formal power series $(a(x)|g(x), f(x))$ with

$$a(x) = a'_0 + a'_1 x + a'_2 x^2 + \dots$$
$$g(x) = g_0 + g_1 x + g_2 x^2 + \dots$$

and

$$f(x) = f_0 + f_1 x + f_2 x^2 + \dots$$

with $a'_0 \neq 0$, $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$ is an almost-Riordan array of order 1. The generating function of the kth column of the almost-Riordan array of order 1 is

$$a(x), \quad for \;\; k = 0 \tag{1.8}$$

$$xg(x)(f(x))^{k-1}, \quad for \;\; k = 1, 2, 3, \dots \tag{1.9}$$

Also, the multiplication of two almost-Riordan arrays is defined as follows

$$(a(x)|g(x), f(x))(b(x)|h(x), l(x)) = ((a(x)|g(x), f(x))b(x)\big|g(x)h(f(x)), l(f(x))) \tag{1.10}$$

where $(a(x)|g(x), f(x))b(x)$ is

$$(a(x)|g(x), f(x))b(x) = b_0 a(x) + xg(x)\frac{b(f(x)) - b_0}{f(x)}. \tag{1.11}$$

The set of the almost-Riordan arrays of order 1 is a group under multiplication with (1.10). The identity element of the almost-Riordan arrays of order 1 is

$$I = (1|1, x) \tag{1.12}$$

and the inverse of the almost-Riordan arrays is given below

$$(a(x)|g(x), f(x))^{-1} = \left(\frac{1}{a'_0}\left(1 - \frac{x}{g(\overline{f}(x))}\frac{a(\overline{f}(x)) - a'_0}{\overline{f}(x)}\right)\middle|\frac{1}{g(\overline{f}(x))}, \overline{f}(x)\right). \tag{1.13}$$

You can find the detailed information about the almost-Riordan arrays of order 1 and higher orders in [2]. In recent years, the pseudo-involutions and involutions in the almost-Riordan arrays have been studied (see [3,14]). In [6], the authors considered decompositions of the Riordan arrays and factorized the Riordan arrays using a new class of matrix decompositions. In the same article is shown that a Riordan matrix can be factorized into almost-Riordan matrices.

Recently, Bang et al. have studied the different operations on the Riordan arrays. Firstly, they defined the sum of the Riordan arrays and obtained the characterizations of this sum of the Riordan arrays. Then, they have given “Der” and “Flip” operations on the Riordan arrays and investigated the applications of these three operations in [1]. Also, the sum of Riordan arrays are mentioned in [11].

Motivated by the above papers, we consider the characterizations of the almost-Riordan arrays of order 1. Then, we introduce the sum of the almost-Riordan arrays and obtain the characterizations of the sum of almost-Riordan arrays. Note that, throughout in this paper, we consider the almost-Riordan array of order 1.

2. The sequence characterization

In this section, we give A, Z and ω-sequences of the almost-Riordan array, the inverse of the almost-Riordan array and the product of two almost-Riordan arrays. Also, we exemplify that the ω, Z and A-sequences characterize the 0th, 1st and following columns, respectively.

Theorem 2.1. *Let $D = (d_{n,k})_{n,k\geq 0}$ be an infinite lower triangular matrix. If D is an almost-Riordan array, then there exist three sequences $A = \{a_0, a_1, a_2, \ldots\}$, $Z = \{z_0, z_1, z_2, \ldots\}$ and $\omega = \{\omega_0, \omega_1, \omega_2, \ldots\}$. These sequences are defined as follows,*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \ldots, \quad k \geq 1, \tag{2.1}$$

$$d_{n+1,1} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \dots \tag{2.2}$$

$$d_{n+1,0} = \omega_0 d_{n,0} + \omega_1 d_{n,1} + \omega_2 d_{n,2} + \dots \tag{2.3}$$

where the coefficients $\{a_0, a_1, a_2, \dots\}$, $\{z_0, z_1, z_2, \dots\}$ *and* $\{\omega_0, \omega_1, \omega_2, \dots\}$ *are known as* A, Z *and* ω*-sequence of the almost-Riordan array* $D = (a(x)|g(x), f(x))$*, respectively.*

Proof. Let $D = (a(x)|g(x), f(x))$ be the almost-Riordan array. Also, let consider $R = \left(b(x)|\frac{g(x)f(x)}{x}, f(x)\right)$ and we define the almost-Riordan array $(c(x)|A(x), B(x))$. Then, we have

$$(c(x)|A(x), B(x)) = (a(x)|g(x), f(x))^{-1} \left(b(x)\middle|\frac{g(x)f(x)}{x}, f(x)\right)$$

or equivalently

$$(a(x)|g(x), f(x))(c(x)|A(x), B(x)) = \left(b(x)\middle|\frac{g(x)f(x)}{x}, f(x)\right).$$

We find

$$g(x)A(f(x)) = \frac{g(x)f(x)}{x} \ \textit{and} \ B(f(x)) = f(x),$$

which implies

$$A(f(x)) = \frac{f(x)}{x} \ \textit{and} \ B(x) = x.$$

Then, we see $d_{n+1,k+1} = r_{n,k}$. So, (2.1) is obtained.

For proof (2.2), let consider $z_0 = \frac{d_{1,1}}{d_{0,0}}$. Then,

$$d_{2,1} = z_0 d_{1,0} + z_1 d_{1,1} \ \textit{or} \ z_1 = \frac{d_{2,1}d_{0,0} - d_{1,1}d_{1,0}}{d_{1,1}d_{0,0}}.$$

In the same way, we can determine uniquely $z_2, z_3, z_4, \dots$.

Similarly, let consider $\omega_0 = \frac{d_{1,0}}{d_{0,0}}$ in (2.3). Now, we get

$$d_{2,0} = \omega_0 d_{1,0} + \omega_1 d_{1,1} \ \textit{or} \ \omega_1 = \frac{d_{2,0}d_{0,0} - d_{1,0}d_{1,0}}{d_{1,1}d_{0,0}}.$$

In the same manner, we can identify $\omega_2, \omega_3, \omega_4, \dots$. □

Theorem 2.2. *Let* $D = (a(x)|g(x), f(x))$ *be an almost-Riordan array. Also,* $A(x)$*,* $Z(x)$ *and* $\omega(x)$ *are the generating functions of* A*,* Z *and* ω*-sequences, respectively. Then, we have*

$$A(x) = \frac{x}{\overline{f}(x)} \tag{2.4}$$

$$Z(x) = \frac{x[g(\overline{f}(x)) - z_0 a(\overline{f}(x))]}{\overline{f}(x) g(\overline{f}(x))} + z_0, \quad z_0 = \frac{g_0}{a_0'} \tag{2.5}$$

$$\omega(x) = \frac{x[a(\overline{f}(x)) - a_0' - \omega_0 \overline{f}(x) a(\overline{f}(x))]}{(\overline{f}(x))^2 g(\overline{f}(x))} + \omega_0, \ \ \omega_0 = \frac{a_1'}{a_0'}. \tag{2.6}$$

Proof. We consider the following equality

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \dots.$$

If we continue with generating functions, we obtain

$$\frac{xg(x)(f(x))^k}{x} = a_0 xg(x)(f(x))^{k-1} + a_1 xg(x)(f(x))^k + \dots.$$

Then,

$$g(x)(f(x))^k = xg(x)(f(x))^{k-1}(a_0 + a_1 f(x) + a_2 (f(x))^2 \dots).$$

The last expression shows that

$$\frac{f(x)}{x} = A(f(x)).$$

Namely, $A(x) = \frac{x}{\overline{f}(x)}$.

Now, we take into account the following equality

$$d_{n+1,1} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \dots.$$

Considering with the generating functions, we get

$$\frac{xg(x) - 0}{x} = z_0 a(x) + z_1 xg(x) + z_2 xg(x) f(x) + z_3 xg(x)(f(x))^2 + \dots.$$

Hence, we have

$$g(x) = z_0 a(x) + xg(x)(z_1 + z_2 f(x) + z_3 (f(x))^2 + \dots).$$

Then,

$$\frac{g(x) - z_0 a(x)}{xg(x)} = \frac{Z(f(x)) - z_0}{f(x)}.$$

If we take $\overline{f}(x)$ instead of x and subtract $Z(x)$, we find (2.5). If we consider constant term of the last equation, we obtain z_0.

Similarly, we consider the following equality

$$d_{n+1,0} = \omega_0 d_{n,0} + \omega_1 d_{n,1} + \omega_2 d_{n,2} + \dots .$$

Using the generating functions, we get

$$\frac{a(x) - a_0'}{x} = \omega_0 a(x) + \omega_1 x g(x) + \omega_2 x g(x) f(x) + \omega_3 x g(x)(f(x))^2 + \dots .$$

Then, we have

$$\frac{a(x) - a_0'}{x} = \omega_0 a(x) + x g(x)(\omega_1 + \omega_2 f(x) + \omega_3 (f(x))^2 + \dots)$$

or

$$\frac{a(x) - a_0' - \omega_0 x a(x)}{x^2 g(x)} = \frac{\omega(f(x)) - \omega_0}{f(x)}.$$

In the last step, if we take $\overline{f}(x)$ in place of x and subtract $\omega(x)$, we obtain (2.6). In the same way, we find ω_0. □

Note that, Barry used the sequences A, Z and ω-to find the production matrix of the almost-Riordan arrays in [2].

Example 2.3. Let us consider the almost-Riordan array $\left(\frac{2-x}{1-x-x^2}|1, x + x^2\right)$. We have

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 4 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 7 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 11 & 0 & 0 & 1 & 3 & 1 & 0 & 0 & \dots \\ 18 & 0 & 0 & 0 & 3 & 4 & 1 & 0 & \dots \\ 29 & 0 & 0 & 0 & 1 & 6 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We see that the column 0th of the above matrix is composed by Lucas numbers and the row sums of the matrix are twice the Fibonacci numbers. Now, we calculate the A, Z and ω-sequences of this almost-Riordan array. Using (2.4), the generating function of the A-sequence is

$$A(x) = \frac{x}{\frac{\sqrt{4x+1}-1}{2}} = \frac{2x}{\sqrt{4x+1} - 1}$$

and the elements of the A-sequence are

$$1, 1, -1, 2, -5, 14, -42, 132, -429, 1430, \ldots.$$

Considering (2.5), we have the following generating function of the Z-sequence

$$\begin{aligned} Z(x) &= \frac{x\left(1-\frac{1}{2}\frac{2-\frac{\sqrt{4x+1}-1}{2}}{1-\frac{\sqrt{4x+1}-1}{2}-\left(\frac{\sqrt{4x+1}-1}{2}\right)^2}\right)}{\frac{\sqrt{4x+1}-1}{2}} + \frac{1}{2} \\ &= \frac{-1}{8(x-1)}(\sqrt{4x+1} - 4x - \sqrt{(4x+1)^3} + 4) \end{aligned}$$

and the elements of the Z-sequence are

$$\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}, -\frac{5}{2}, \frac{5}{2}, -\frac{23}{2}, \frac{61}{2}, \ldots.$$

If we use (2.6), we obtain the following generating function of the ω-sequence

$$\begin{aligned} \omega(x) &= \frac{x\left(\left(\frac{2-\frac{\sqrt{4x+1}-1}{2}}{1-\frac{\sqrt{4x+1}-1}{2}-\left(\frac{\sqrt{4x+1}-1}{2}\right)^2}\right)\left(1-\frac{1}{2}\frac{\sqrt{4x+1}-1}{2}\right)-2\right)}{\left(\frac{\sqrt{4x+1}-1}{2}\right)^2} + \frac{1}{2} \\ &= \frac{-4x-1}{2(x-1)} \end{aligned}$$

and a few elements of ω-sequence are

$$\frac{1}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \ldots.$$

Example 2.4. We define an almost-Riordan array with $D = (d_{n,k}) = (a(x)|g(x), f(x))$. We consider the following equalities,

$$d_{0,0} = 1, \;\; d_{0,k} = 0 \;\; for \;\; k \geq 1, \;\; d_{n+1,0} = abd_{n,0} \tag{2.7}$$

$$d_{1,1} = 1, \;\; d_{1,k} = 0 \;\; for \;\; k \geq 2, \;\; d_{n+1,1} = d_{n,0} \tag{2.8}$$

$$d_{n+1,k+1} = b^2 d_{n,k} + abd_{n,k+1} \;\; for \;\; k \geq 1 \tag{2.9}$$

where a and b are any two nonzero real variables. Using (2.7), (2.8) and (2.9), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ ab & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a^2b^2 & ab & b^2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a^3b^3 & a^2b^2 & 2ab^3 & b^4 & 0 & 0 & 0 & 0 & \cdots \\ a^4b^4 & a^3b^3 & 3a^2b^4 & 3ab^5 & b^6 & 0 & 0 & 0 & \cdots \\ a^5b^5 & a^4b^4 & 4a^3b^5 & 6a^2b^6 & 4ab^7 & b^8 & 0 & 0 & \cdots \\ a^6b^6 & a^5b^5 & 5a^4b^6 & 10a^3b^7 & 10a^2b^8 & 5ab^9 & b^{10} & 0 & \cdots \\ a^7b^7 & a^6b^6 & 6a^5b^7 & 15a^4b^8 & 20a^3b^9 & 15a^2b^{10} & 6ab^{11} & b^{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now, we find the representation of the almost-Riordan array D. Then, we get

$$\begin{aligned} a(x) &= 1 + abx + a^2b^2x^2 + a^3b^3x^3 + \dots \\ a(x) &= 1 + abx(1 + abx + a^2b^2x^2 + \dots) \\ a(x) &= 1 + abxa(x) \\ a(x) &= \frac{1}{1-abx}. \end{aligned}$$

Now, we obtain $g(x)$,

$$\begin{aligned} xg(x) &= x + abx^2 + a^2b^2x^3 + \dots \\ xg(x) &= x(1 + abx + a^2b^2x^2 + \dots) \\ g(x) &= 1 + abx + a^2b^2x^2 + \dots \\ g(x) &= \frac{1}{1-abx}. \end{aligned}$$

Using (2.9), we have

$$\begin{aligned} \frac{xg(x)(f(x))^k}{x} &= b^2xg(x)(f(x))^{k-1} + abxg(x)(f(x))^k \\ g(x)(f(x))^k &= (b^2 + abf(x))xg(x)(f(x))^{k-1} \\ f(x) &= \frac{b^2x}{1-abx}. \end{aligned}$$

Finally, the almost-Riordan array D is represented with

$$D = \left(\frac{1}{1-abx} \middle| \frac{1}{1-abx}, \frac{b^2x}{1-abx} \right).$$

Now, we give the A, Z and ω-sequences for the inverse of an almost-Riordan array.

Theorem 2.5. *Let $D = (a(x)|g(x), f(x))$ be an almost-Riordan array and $D^{-1} = (a(x)|g(x), f(x))^{-1}$ be the inverse of the almost-Riordan array D. $A(x)$ and $A^*(x)$ are the generating functions of the A-sequence for D and D^{-1}, respectively. Then, we get*

$$A^*(x) = \frac{1}{A(f(x))}. \tag{2.10}$$

Proof. If we consider (1.13) and (2.4), we have

$$A^*(x) = \frac{x}{\overline{\overline{(f}(x))}} = \frac{x}{f(x)} = \frac{1}{A(f(x))}. \qquad \square$$

Theorem 2.6. *Let $D = (a(x)|g(x), f(x))$ be an almost-Riordan array and $D^{-1} = (a(x)|g(x), f(x))^{-1}$ be the inverse of the almost-Riordan array D. The generating function for the Z-sequence of D^{-1} is*

$$Z^*(x) = A^*(x) + g(x)\frac{a_0'}{g_0^2}\left(1 - Z\left(\frac{x}{A^*(x)}\right)A^*(x)\right) \tag{2.11}$$

where $A^*(x)$ is the generating function of the A-sequence of D^{-1} and $Z(x)$ and $Z^*(x)$ are the generating functions of the Z-sequence for D and D^{-1}, respectively.

Proof. Using (1.13) and (2.5), we find

$$\begin{aligned} Z^*(\overline{f}(x)) &= \frac{\overline{f}(x)\frac{1}{g(\overline{f}(x))} - \overline{f}(x)z_0^*\frac{1}{a_0'}\left(1 - \frac{xa(\overline{f}(x)) - xa_0'}{g(\overline{f}(x))\overline{f}(x)}\right)}{x\frac{1}{g(\overline{f}(x))}} + z_0^* \\ &= \frac{a_0'\overline{f}(x) - z_0^*\overline{f}(x)g(\overline{f}(x)) + z_0^*xa(\overline{f}(x)) - z_0^*a_0'x}{a_0'x} + z_0^*. \end{aligned}$$

Taking $x = f(x)$ and $A^*(x) = \frac{x}{f(x)}$, we have

$$Z^*(x) = A^*(x) + \frac{z_0^*}{a_0'}\left(a(x) - A^*(x)g(x)\right).$$

In this step, considering the inverse of the almost-Riordan array D^{-1}, we find that

$$z_0^* = \frac{a_0'}{g_0}. \tag{2.12}$$

Taking $x = f(x)$ in (2.5) and subtracting $a(x)$ from it, we have

$$a(x) = \frac{xg(x)}{z_0 f(x)}\left(\frac{f(x)}{x} + z_0 - Z(f(x))\right). \tag{2.13}$$

Finally, using (2.10), (2.12) and (2.13), we obtain the result. $\square$

Theorem 2.7. *Let $D = (a(x)|g(x), f(x))$ be an almost-Riordan array and $D^{-1} = (a(x)|g(x), f(x))^{-1}$ be the inverse of the almost-Riordan array D. $\omega(x)$ and $\omega^*(x)$ are the generating functions of the ω-sequences of D and D^{-1}, respectively. Then, we have*

$$\omega^*(x) = \frac{1}{a_0' - a_1'x}\begin{pmatrix} a_1'g(x)A^*(x)\left(\frac{1}{a_0'}A^*(x) + \frac{1}{g_0}\right) - a_1'A^*(x) \\ -g(x)A^*(x)\omega\left(\frac{x}{A^*(x)}\right)\left(A^*(x) + \frac{a_1'}{g_0}x\right) - \frac{a_0'a_1'}{g_0} \end{pmatrix} \tag{2.14}$$

where $A^*(x)$ is the generating function of the A-sequence for D^{-1}.

Proof. If we use (1.13) and (2.6), we have

$$\omega^*(\overline{f}(x)) = \frac{\overline{f}(x)\left(\frac{1}{a_0'}\left(1 + \frac{a_0'x - xa(\overline{f}(x))}{\overline{f}(x)g(\overline{f}(x))}\right)(1 - x\omega_0^*) - \frac{1}{a_0'}\right)}{\frac{x^2}{g(\overline{f}(x))}} + \omega_0^*.$$

Considering $DD^{-1} = I$, we find that

$$\omega_0^* = -\frac{a_1'}{g_0}.$$

Then, we get

$$\begin{aligned} \omega^*(\overline{f}(x)) &= \begin{pmatrix} \frac{\overline{f}(x)g(\overline{f}(x))}{a_0'x^2} - \frac{a(\overline{f}(x))}{a_0'x} + \frac{1}{x} + \frac{a_1'\overline{f}(x)g(\overline{f}(x))}{a_0'g_0x} \\ -\frac{a_1'a(\overline{f}(x))}{a_0'g_0} + \frac{a_1'}{g_0} - \frac{g(\overline{f}(x))\overline{f}(x)}{a_0'x^2} - \frac{a_1'}{g_0} \end{pmatrix} \\ &= \frac{a(\overline{f}(x))}{a_0'}\left(-\frac{1}{x} - \frac{a_1'}{g_0}\right) + \frac{1}{x}\left(1 + \frac{a_1'\overline{f}(x)g(\overline{f}(x))}{a_0'g_0}\right). \end{aligned}$$

If we take $x = f(x)$ in (2.6) and subtract $a(x)$ from it, we get

$$a(x) = \frac{\frac{x^2g(x)}{f(x)}\left(\omega(f(x)) - \frac{a_1'}{a_0'}\right) + a_0'}{1 - \frac{a_1'}{a_0'}x}. \tag{2.15}$$

Using (2.10) and (2.15), the result is obtained. □

Example 2.8. We consider the almost-Riordan array $D = \left(\frac{2-x}{1-x-x^2}\middle|\frac{3}{1-x-x^2}, \frac{x}{1-x}\right)$. Namely

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 6 & 6 & 3 & 0 & 0 & 0 & 0 & \cdots \\ 7 & 9 & 12 & 9 & 3 & 0 & 0 & 0 & \cdots \\ 11 & 15 & 21 & 21 & 12 & 3 & 0 & 0 & \cdots \\ 18 & 24 & 36 & 42 & 33 & 15 & 3 & 0 & \cdots \\ 29 & 39 & 60 & 78 & 75 & 48 & 18 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Its inverse is

$$D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{3} & 0 & -\frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \dots \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} & -1 & \frac{1}{3} & 0 & 0 & 0 & \dots \\ \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} & \frac{5}{3} & -\frac{4}{3} & \frac{1}{3} & 0 & 0 & \dots \\ -\frac{1}{3} & 1 & -\frac{1}{3} & -2 & 3 & -\frac{5}{3} & \frac{1}{3} & 0 & \dots \\ \frac{1}{3} & -\frac{4}{3} & \frac{4}{3} & \frac{5}{3} & -5 & \frac{14}{3} & -2 & \frac{1}{3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The generating functions of the A, Z and ω-sequences for the almost-Riordan array D are as follows

$$A(x) = 1 + x \tag{2.16}$$

$$Z(x) = \tfrac{3}{2} + \tfrac{1}{2}x \tag{2.17}$$

$$\omega(x) = \tfrac{1}{2} + \tfrac{5}{6}x. \tag{2.18}$$

Now, let us calculate the A^*, Z^* and ω^*-sequences of the almost-Riordan array D^{-1}. Using (2.10) and (2.16), we have

$$A^*(x) = 1 - x. \tag{2.19}$$

From here, the first few elements of the A^*-sequence are

$$1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots .$$

Using (2.11), (2.17) and (2.19), we get the generating function of the Z^*-sequence

$$\begin{aligned} Z^*(x) &= 1 - x + \frac{2}{3(1-x-x^2)}\left(1 - (1-x)\left(\frac{3}{2} + \frac{x}{2(1-x)}\right)\right) \\ &= \frac{3x^3 - 4x + 2}{3(1-x-x^2)} \end{aligned}$$

and the first terms of the Z^*-sequence are

$$\frac{2}{3}, -\frac{2}{3}, 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 1, \frac{5}{3}, \frac{8}{3}, \frac{13}{3}, 7, \frac{34}{3}, \dots .$$

Finally, using (2.14), (2.18) and (2.19), we obtain the generating function of the ω^*-sequence as follows

$$\omega^*(x) = \frac{1}{2-x}\left(\begin{array}{l} \frac{3}{1-x-x^2}(1-x)\left(\frac{1}{2} + \frac{5x}{6(1-x)}\right)\left(-1 + x - \frac{1}{3}x\right) \\ + \frac{3}{1-x-x^2}(1-x)\left(\frac{1}{2}(1-x) + \frac{1}{3}\right) - (1-x) - \frac{2}{3} \end{array}\right)$$

$$= \frac{-6x^2 + 5x + 2}{6(x^2 + x - 1)}$$

and the first few elements of the ω^*-sequence are

$$-\frac{1}{3}, -\frac{7}{6}, -\frac{1}{2}, -\frac{5}{3}, -\frac{13}{6}, -\frac{23}{6}, -6, -\frac{59}{6}, \ldots.$$

In this part of the study, we obtain the A, Z and ω-sequences of the product of two almost-Riordan arrays.

Theorem 2.9. *We consider the two almost-Riordan arrays* $D_1 = (a(x)|g(x), f(x))$ *and* $D_2 = (b(x)|h(x), l(x))$ *and their product* $D_3 = (c(x)|m(x), n(x))$. *Also, their A-sequences are represented with* A_1, A_2 *and* A_3, *respectively. Then, we have*

$$A_3(x) = A_1\left(\frac{x}{A_2(x)}\right) A_2(x). \tag{2.20}$$

Proof. Using (2.4), we get

$$A_3(x) = \frac{x}{\overline{n}(x)}. \tag{2.21}$$

Also, considering the product of two almost-Riordan arrays, we obtain

$$\begin{aligned} n(x) &= l(f(x)) \\ \bar{l}(n(x)) &= f(x) \\ \overline{f}(\bar{l}(n(x))) &= x \\ \overline{f}(\bar{l}(x)) &= \overline{n}(x). \end{aligned} \tag{2.22}$$

Using (2.4), (2.21), (2.22) and $A_2(x) = \frac{x}{\bar{l}(x)}$, we have

$$A_3(x) = \frac{x}{\overline{f}(\bar{l}(x))} = \frac{x}{\overline{f}\left(\frac{x}{A_2(x)}\right)}.$$

Using $A_1(x) = \frac{x}{\overline{f}(x)}$, we get

$$A_3(x) = A_1\left(\frac{x}{A_2(x)}\right) A_2(x). \quad \square$$

Theorem 2.10. *We consider the two almost-Riordan arrays* $D_1 = (a(x)|g(x), f(x))$ *and* $D_2 = (b(x)|h(x), l(x))$ *and their product* $D_3 = (c(x)|m(x), n(x))$. *The* Z_3*-sequence of* D_3 *is*

$$Z_3(x) = \frac{h_0}{h\left(\frac{x}{A_2(x)}\right)} \left(\frac{A_3(x)}{A_1\left(\frac{x}{A_2(x)}\right)} \left(Z_1\left(\frac{x}{A_2(x)}\right) - \frac{g_0}{a_0'} \right) - A_3(x) + \frac{g_0}{a_0'} A_2(x) \right) + A_3(x) + \frac{g_0}{a_0'}(Z_2(x) - A_2(x)) \tag{2.23}$$

where Z_1 and Z_2 are the Z-sequences of D_1 and D_2, respectively.

Proof. Considering (1.10), (1.11) and (2.5), we have

$$\begin{aligned} Z_3(x) &= \frac{x\left(m(\overline{n}(x)) - \frac{m_0}{c_0} c(\overline{n}(x))\right)}{\overline{n}(x) m(\overline{n}(x))} + \frac{m_0}{c_0} \\ &= \frac{x}{\overline{n}(x)} + \frac{g_0 h_0}{a_0' b_0} - \frac{g_0 h_0}{a_0'} \frac{x a(\overline{n}(x))}{\overline{n}(x) g(\overline{n}(x)) h(\bar{l}(x))} \\ &+ \frac{g_0 h_0}{a_0'} \frac{x}{h(\bar{l}(x))\bar{l}(x)} - \frac{g_0 h_0}{a_0' b_0} \frac{x b(\bar{l}(x))}{\bar{l}(x) h(\bar{l}(x)))}. \end{aligned}$$

In this step, using the following equalities

$$\begin{aligned} \frac{b(\bar{l}(x))}{h(\bar{l}(x))} &= \frac{b_0}{h_0 A_2(x)} \left(A_2(x) - Z_2(x) + \frac{h_0}{b_0} \right) \\ \frac{a(\overline{n}(x))}{g(\overline{n}(x))} &= \frac{a_0'}{g_0 A_1(\bar{l}(x))} \left(A_1(\bar{l}(x)) - Z_1(\bar{l}(x)) + \frac{g_0}{a_0'} \right) \end{aligned}$$

and $A_1(x) = \frac{x}{\bar{f}(x)}$, $A_2(x) = \frac{x}{\bar{l}(x)}$, $A_3(x) = \frac{x}{\overline{n}(x)}$, we have

$$\begin{aligned} Z_3(x) &= A_3(x) + \frac{g_0}{a_0'}(Z_2(x) - A_2(x)) \\ &+ \frac{h_0 g_0}{a_0' h\left(\frac{x}{A_2(x)}\right)} \left(A_2(x) - \frac{A_3(x)}{A_1\left(\frac{x}{A_2(x)}\right)} \right) \\ &+ \frac{h_0}{h\left(\frac{x}{A_2(x)}\right)} \left(Z_1\left(\frac{x}{A_2(x)}\right) \frac{A_3(x)}{A_1\left(\frac{x}{A_2(x)}\right)} - A_3(x) \right). \end{aligned}$$

By simplifying, this expression is equivalent to (2.23). □

Theorem 2.11. *Consider the two almost-Riordan arrays $D_1 = (a(x)|g(x), f(x))$ and $D_2 = (b(x)|h(x), l(x))$ and their product $D_3 = (c(x)|m(x), n(x))$. Then, the following identity holds*

$$\omega_3(x) = \frac{1}{h\left(\frac{x}{A_2(x)}\right)} \left(\begin{array}{c} \frac{a_0' A_3(x)}{g_0 A_1\left(\frac{x}{A_2(x)}\right)} \left(A_1\left(\frac{x}{A_2(x)}\right) - Z_1\left(\frac{x}{A_2(x)}\right) + \frac{g_0}{a_0'}\right)\left(b_0 \frac{A_3(x)}{x} - \frac{a_1' b_0 + g_0 b_1}{a_0'}\right) \\ -A_3(x)\left(b_0 \frac{A_2(x)}{x} + a_0' b_0 \frac{A_3(x)}{xg\left(\frac{x}{A_3(x)}\right)}\right) + \frac{a_1' b_0 + g_0 b_1}{a_0'} A_2(x) \end{array} \right)$$
$$+ \frac{b_0}{h_0}\left(A_2(x) - Z_2(x) + \frac{h_0}{b_0}\right)\left(\frac{1}{x}A_3(x) - \frac{a_1' b_0 + g_0 b_1}{a_0' b_0}\right)$$
$$+ \frac{a_1' b_0 + g_0 b_1}{a_0' b_0} \tag{2.24}$$

where $\omega_3(x)$ is the generating function for the ω-sequence of D_3.

Proof. Using (1.10), (1.11) and (2.6), we get

$$\begin{aligned} \omega_3(x) &= \frac{x\left(c(\overline{n}(x))\left(1 - \frac{c_1}{c_0}\overline{n}(x)\right) - c_0\right)}{(\overline{n}(x))^2 m(\overline{n}(x))} + \frac{c_1}{c_0} \\ &= \frac{b_0 x\left(a(\overline{n}(x)) - a_0'\right)}{(\overline{n}(x))^2 g(\overline{n}(x)) h(\overline{l}(x))} - \frac{a_1' b_0 + g_0 b_1}{a_0'} \frac{x a(\overline{n}(x))}{\overline{n}(x) g(\overline{n}(x)) h(\overline{l}(x))} \\ &+ \frac{x\left(b(\overline{l}(x)) - b_0\right)}{\overline{l}(x)\overline{n}(x) h(\overline{l}(x))} + \frac{(a_1' b_0 + g_0 b_1)x}{a_0' h(\overline{l}(x))\overline{l}(x)} \\ &- \frac{a_1' b_0 + g_0 b_1}{a_0' b_0} \frac{x b(\overline{l}(x))}{\overline{l}(x) h(\overline{l}(x))} + \frac{a_1' b_0 + g_0 b_1}{a_0' b_0}. \end{aligned}$$

Then, we have

$$\begin{aligned} \omega_3(x) &= \frac{a_0' A_3(x)}{g_0 h(\overline{l}(x)) A_1(\overline{l}(x))}\left(A_1(\overline{l}(x)) - Z_1(\overline{l}(x)) + \frac{g_0}{a_0'}\right)\left(\frac{b_0}{\overline{n}(x)} - \frac{a_1' b_0 + g_0 b_1}{a_0' b_0}\right) \\ &+ \frac{a_1' b_0 + g_0 b_1}{a_0' b_0} + \frac{1}{h(\overline{l}(x))}\left(\frac{a_1' b_0 + g_0 b_1}{a_0'} A_2(x) - A_3(x)\left(\frac{b_0}{\overline{l}(x)} + \frac{a_0' b_0}{\overline{n}(x) g(\overline{n}(x))}\right)\right) \\ &+ \frac{b_0}{h_0 A_2(x)}\left(A_2(x) - Z_2(x) + \frac{h_0}{b_0}\right)\left(\frac{A_3(x)}{\overline{l}(x)} - \frac{a_1' b_0 + g_0 b_1}{a_0' b_0} A_2(x)\right). \end{aligned}$$

If we use $A_2(x) = \frac{x}{\overline{l}(x)}$ and $A_3(x) = \frac{x}{\overline{n}(x)}$ in the last step, we obtain the result. □

Example 2.12. Let D_1 and D_2 be the almost-Riordan arrays as follows

$$D_1 = \left(\frac{1 - x + x^2}{1 - 2x + x^3}\middle| \frac{1}{1 - x - x^2}, x\right) \quad \textit{and} \quad D_2 = \left(1 \middle| \frac{1 - \sqrt{1 - 4x}}{2x}, \frac{1 - \sqrt{1 - 4x}}{2}\right).$$

From (1.10), we have

$$D_3 = D_1 D_2 = \left(\frac{1 - x + x^2}{1 - 2x + x^3}\middle| \frac{1 - \sqrt{1 - 4x}}{2x(1 - x - x^2)}, \frac{1 - \sqrt{1 - 4x}}{2}\right).$$

Namely,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 3 & 2 & 1 & 0 & 0 & 0 & \dots \\ 5 & 5 & 3 & 1 & 0 & 0 & \dots \\ 9 & 12 & 9 & 4 & 1 & 0 & \dots \\ 15 & 31 & 26 & 14 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 3 & 1 & 1 & 0 & 0 & 0 & \dots \\ 5 & 2 & 1 & 1 & 0 & 0 & \dots \\ 9 & 3 & 2 & 1 & 1 & 0 & \dots \\ 15 & 5 & 3 & 2 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 2 & 1 & 0 & 0 & \dots \\ 0 & 5 & 5 & 3 & 1 & 0 & \dots \\ 0 & 14 & 14 & 9 & 4 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The generating functions of the A, Z and ω-sequences of D_1 and D_2 are

$$A_1(x) = 1 \tag{2.25}$$

$$Z_1(x) = \frac{1 - x - x^2}{1 - x} \tag{2.26}$$

$$\omega_1(x) = 1 + 2x \tag{2.27}$$

and

$$A_2(x) = \frac{1}{1 - x} \tag{2.28}$$

$$Z_2(x) = \frac{1}{1 - x} \tag{2.29}$$

$$\omega_2(x) = 0. \tag{2.30}$$

Now, we find the A_3, Z_3 and ω_3-sequences of D_3. If we use (2.20), (2.25) and (2.28), we obtain the generating function of A_3-sequence as follows

$$A_3(x) = \frac{1}{1 - x} \tag{2.31}$$

and the first few elements of A_3-sequence are

$$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \dots.$$

Using (2.23), (2.25), (2.26), (2.28), (2.29) and (2.31), we get the generating function of the Z_3-sequence

$$Z_3(x) = \frac{1}{1-x} + \frac{2x(1-x)}{1-\sqrt{1-4x(1-x)}}\frac{1}{1-x}\left(\frac{1-(x-x^2)-(x-x^2)^2}{1-(x-x^2)} - 1\right)$$
$$= \frac{x^5-3x^4+3x^3-x+1}{-x^3+2x^2-2x+1}$$

and the first terms of the Z_3-sequence are

$$1, 1, 0, 2, 2, 1, 0, 0, 1, 2, 2, 1, \ldots .$$

Using (2.24), (2.25), (2.26), (2.28), (2.29) and (2.31), the generating function of the ω_3-sequence is as follows

$$\omega_3(x) = \frac{2x(1-x)}{1-\sqrt{1-4x(1-x)}}\frac{1}{1-x}\left(2 - \frac{1-(x-x^2)-(x-x^2)^2}{1-(x-x^2)}\right)\left(\frac{1}{x(1-x)} - 1\right)$$
$$- \frac{2x(1-x)}{1-\sqrt{1-4x(1-x)}}\left(\frac{1}{x(1-x)^2}\left(2-(x-x^2)-(x-x^2)^2\right) - \frac{1}{1-x}\right)$$
$$+ \frac{1}{x(1-x)}$$
$$= -2x^2+2x+1.$$

The first few elements of the ω_3-sequence are

$$1, 2, -2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots .$$

3. Sum of the almost-Riordan arrays

In this part of the study, the sum of almost-Riordan arrays is investigated. In general, the sum of two almost-Riordan arrays can not be the almost-Riordan array. The following theorem shows under what conditions the sum of two almost-Riordan arrays is again the almost-Riordan array.

Theorem 3.1. *Let $D_1 = (a(x)|g(x), f(x))$ and $D_2 = (b(x)|h(x), l(x))$ be two almost-Riordan arrays. Then $D_1 + D_2$ is an almost-Riordan array if and only if $f(x) = l(x)$, $a_0' + b_0 \neq 0$ and $g_0 + h_0 \neq 0$. Then, we have*

$$D_1 + D_2 = (a(x) + b(x)|g(x) + h(x), f(x)). \tag{3.1}$$

Proof. Let $D_1 + D_2 = (c(x)|m(x), n(x))$ be an almost-Riordan array and $C_i(x)$ be the generating function of ith column of the almost-Riordan array $D_1 + D_2$.

$$C_0(x) = c(x) = a(x) + b(x).$$

We see that $a_0' + b_0 \neq 0$ because of $c_0 \neq 0$. Considering the matrix equality, we get

$$\begin{aligned}C_i(x) &= xm(x)(n(x))^{i-1}\\ &= xg(x)(f(x))^{i-1} + xh(x)(l(x))^{i-1},\end{aligned}$$

for $i \geq 1$. Since $D_1 + D_2$ is the almost-Riordan array, we see that $n(x) = f(x) = l(x)$ and $g_0 + h_0 \neq 0$.

Conversely, let $f(x) = l(x)$, $a'_0 + b_0 \neq 0$ and $g_0 + h_0 \neq 0$ equalities be satisfied. Then, let us show that D_1+D_2 is an almost-Riordan array. $C_0(x) = a(x)+b(x)$ and $a'_0+b_0 \neq 0$, we have $c_0 \neq 0$. So, the first column satisfies the almost-Riordan array conditions. Then,

$$\begin{aligned}C_i(x) &= xg(x)(f(x))^{i-1} + xh(x)(l(x))^{i-1}\\ &= xg(x)(f(x))^{i-1} + xh(x)(f(x))^{i-1}\\ &= x(g(x) + h(x))(f(x))^{i-1}.\end{aligned}$$

From the above equalities, $D_1 + D_2$ is an almost-Riordan array. □

Theorem 3.2. *We consider the following subset of the almost-Riordan arrays*

$$G_{f(x)} = \{(a_i(x)|g_i(x), f(x)) : a_i(0) + a_j(0) \neq 0, g_i(0) + g_j(0) \neq 0, i, j = 0, 1, \ldots\}.$$

Also, we consider $(0|0, f(x))$ *which is not an almost-Riordan array. Then, the set* $G_{f(x)} \cup (0|0, f(x))$ *is a commutative monoid with addition operation.*

Now, we give the A, Z and ω-sequences of the sum of two almost-Riordan arrays. In the following theorems, we assume that $D_1 = (a(x)|g(x), f(x))$, $D_2 = (b(x)|h(x), f(x))$, $a'_0 + b_0 \neq 0, g_0 + h_0 \neq 0$ and $D_1 + D_2 = D_3$.

Theorem 3.3. *Let* $Z_3(x)$ *be the generating function of the* Z*-sequence for* D_3*. Then, we have*

$$\begin{aligned}Z_3(x) = {} & \frac{g_0 + h_0}{(a'_0 + b_0)\left(g\left(\frac{x}{A(x)}\right) + h\left(\frac{x}{A(x)}\right)\right)} \begin{pmatrix} \frac{a'_0}{g_0} Z_1(x) g\left(\frac{x}{A(x)}\right) \\ + \frac{b_0}{h_0} Z_2(x) h\left(\frac{x}{A(x)}\right)\end{pmatrix} \\ & + \frac{(b_0 g_0 - a'_0 h_0) A(x)\left(h_0 g\left(\frac{x}{A(x)}\right) - g_0 h\left(\frac{x}{A(x)}\right)\right)}{g_0 h_0 (a'_0 + b_0)\left(g\left(\frac{x}{A(x)}\right) + h\left(\frac{x}{A(x)}\right)\right)}\end{aligned} \tag{3.2}$$

where $Z_1(x)$ *and* $Z_2(x)$ *are the generating functions of the* Z*-sequences for* D_1 *and* D_2*, respectively.*

Proof. The generating functions for the 0th columns of D_1 and D_2 are $a(x)$ and $b(x)$, respectively. Also, the generating function for the 0th column of $D_1 + D_2$ is $c(x) = a(x) + b(x)$. Taking $x = f(x)$ in (2.5), we have $a(x), b(x)$ and $c(x)$ as follows,

$$a(x) = -\frac{a_0'}{g_0} g(x) \left(\frac{x}{f(x)} \left(Z_1(f(x)) - \frac{g_0}{a_0'} \right) - 1 \right)$$
$$b(x) = -\frac{b_0}{h_0} h(x) \left(\frac{x}{f(x)} \left(Z_2(f(x)) - \frac{h_0}{b_0} \right) - 1 \right)$$

and

$$c(x) = -\frac{a_0' + b_0}{g_0 + h_0} (g(x) + h(x)) \left(\frac{x}{f(x)} \left(Z_3(f(x)) - \frac{g_0 + h_0}{a_0' + b_0} \right) - 1 \right).$$

Considering $c(x) = a(x) + b(x)$, we get

$$-\frac{(a_0' + b_0)(g(x) + h(x))}{g_0 + h_0} \left(\frac{x}{f(x)} \left(Z_3(f(x)) - \frac{g_0 + h_0}{a_0' + b_0} \right) - 1 \right)$$
$$= \begin{pmatrix} -\frac{a_0'}{g_0} g(x) \left(\frac{x}{f(x)} \left(Z_1(f(x)) - \frac{g_0}{a_0'} \right) - 1 \right) \\ -\frac{b_0}{h_0} h(x) \left(\frac{x}{f(x)} \left(Z_2(f(x)) - \frac{h_0}{b_0} \right) - 1 \right) \end{pmatrix}$$

Hence, the left hand side equals

$$\frac{x}{f(x)} (g(x) + h(x)) \left(-\frac{a_0' + b_0}{g_0 + h_0} Z_3(f(x)) + 1 \right) + \frac{a_0' + b_0}{g_0 + h_0} (g(x) + h(x)).$$

Also, the right hand side equals

$$\frac{xg(x)}{f(x)} \left(-\frac{a_0'}{g_0} Z_1(f(x)) + 1 \right) + \frac{xh(x)}{f(x)} \left(-\frac{b_0}{h_0} Z_1(f(x)) + 1 \right) + \frac{a_0'}{g_0} g(x) + \frac{b_0}{h_0} h(x).$$

Its follow that

$$\frac{a_0' + b_0}{g_0 + h_0} \frac{x}{f(x)} (g(x) + h(x)) Z_3(f(x)) = \frac{x}{f(x)} \left(\frac{a_0'}{g_0} g(x) Z_1(f(x)) + \frac{b_0}{h_0} h(x) Z_2(f(x)) \right)$$
$$- \frac{(a_0' h_0 - b_0 g_0)(h_0 g(x) - g_0 h(x))}{g_0 h_0 (g_0 + h_0)}.$$

Then,

$$Z_3(f(x)) = \frac{g_0 + h_0}{a_0' + b_0} \frac{1}{g(x) + h(x)} \left(\frac{a_0'}{g_0} g(x) Z_1(f(x)) + \frac{b_0}{h_0} h(x) Z_2(f(x)) \right)$$
$$- \frac{(a_0' h_0 - b_0 g_0)(h_0 g(x) - g_0 h(x)) f(x)}{g_0 h_0 (a_0' + b_0) x (g(x) + h(x))}.$$

Taking $x = \overline{f}(x)$ and $A(x) = \frac{x}{\overline{f}(x)}$, the desired result is obtained. □

Theorem 3.4. *Let $\omega_3(x)$ be the generating function of the ω-sequence for D_3. Then, we have*

$$\omega_3(x) = \frac{1}{a'_0+b_0}\left(\frac{\left[\left(a\left(\frac{x}{A(x)}\right)+b\left(\frac{x}{A(x)}\right)\right)\left(a'_0+b_0-(a'_1+b_1)\frac{x}{A(x)}\right)-(a'_0+b_0)^2\right](a'_0\omega_1(x)-a'_1)(b_0\omega_2(x)-b_1)}{(b_0\omega_2(x)-b_1)\left[a\left(\frac{x}{A(x)}\right)\left(a'_0-a'_1\frac{x}{A(x)}\right)-(a'_0)^2\right]+(a'_0\omega_1(x)-a'_1)\left[b\left(\frac{x}{A(x)}\right)\left(b_0-b_1\frac{x}{A(x)}\right)-b_0^2\right]}\right) + \frac{a'_1+b_1}{a'_0+b_0} \tag{3.3}$$

where $\omega_1(x)$ and $\omega_2(x)$ are the generating functions of ω-sequences for D_1 and D_2, respectively.

Proof. Let $D_1 + D_2 = D_3 = (c(x)|m(x), n(x))$ be an almost-Riordan array. The generating functions for the 1th columns of D_1 and D_2 are $g(x)$ and $h(x)$, respectively. Also, the generating function for the 1th column of $D_1 + D_2$ is $m(x) = g(x) + h(x)$. Taking $x = f(x)$ in (2.6), we have $g(x), h(x)$ and $m(x)$ as follows,

$$g(x) = \frac{a(x)\left(1-\frac{a'_1}{a'_0}x\right)-a'_0}{x^2\left(\omega_1(f(x))-\frac{a'_1}{a'_0}\right)}$$

$$h(x) = \frac{b(x)\left(1-\frac{b_1}{b_0}x\right)-b_0}{x^2\left(\omega_2(f(x))-\frac{b_1}{b_0}\right)}$$

and

$$m(x) = \frac{(a(x)+b(x))\left(1-\frac{a'_1+b_1}{a'_0+b_0}x\right)-a'_0-b_0}{x^2\left(\omega_3(f(x))-\frac{a'_1+b_1}{a'_0+b_0}\right)}.$$

Then, we get

$$\frac{(a(x)+b(x))\left(1-\frac{a'_1+b_1}{a'_0+b_0}x\right)-a'_0-b_0}{x^2\left(\omega_3(f(x))-\frac{a'_1+b_1}{a'_0+b_0}\right)} = \frac{a(x)\left(1-\frac{a'_1}{a'_0}x\right)-a'_0}{x^2\left(\omega_1(f(x))-\frac{a'_1}{a'_0}\right)} + \frac{b(x)\left(1-\frac{b_1}{b_0}x\right)-b_0}{x^2\left(\omega_2(f(x))-\frac{b_1}{b_0}\right)}.$$

Hence, we have

$$\begin{aligned}&\frac{(a(x)+b(x))(a'_0+b_0-x(a'_1+b_1))-(a'_0+b_0)^2}{(a'_0+b_0)\omega_3(f(x))-a'_1-b_1}\\ &= \frac{(b_0\omega_2(f(x))-b_1)(a(x)(a'_0-a'_1x)-(a'_0)^2)}{(a'_0\omega_1(f(x))-a'_1)(b_0\omega_2(f(x))-b_1)}\\ &\quad + \frac{(a'_0\omega_1(f(x))-a'_1)(b(x)(b_0-b_1x)-b_0^2)}{(a'_0\omega_1(f(x))-a'_1)(b_0\omega_2(f(x))-b_1)}.\end{aligned}$$

Also, the right hand side equals

$$[(a_0'+b_0)\omega_3(f(x))-a_1'-b_1]\begin{bmatrix}(b_0\omega_2(f(x))-b_1)(a_0'a(x)-(a_0')^2-a_1'xa(x))\\+(a_0'\omega_1(f(x))-a_1')(b_0b(x)-b_0^2-b_1xb(x))\end{bmatrix}.$$

The left hand side equals

$$(a(x)+b(x))(a_0'+b_0-(a_1'+b_1)x-(a_0'+b_0)^2)(a_0'\omega_1(f(x))-a_1')(b_0\omega_2(f(x))-b_1).$$

In the last step, taking $x=\overline{f}(x)$ and $A(x)=\frac{x}{\overline{f}(x)}$, we obtain the result. □

Note that the generating functions of the A-sequence for D_1, D_2 and D_3 is the same with

$$A(x)=\frac{x}{\overline{f}(x)}. \tag{3.4}$$

Example 3.5. Let D_1 and D_2 be the almost-Riordan arrays as follows

$$D_1=\left(\frac{1}{1-x-x^2}\middle|\frac{1+x-x^2}{1-2x+x^3},x\right)\ \ and\ \ D_2=\left(\frac{2-x}{1-x-x^2}\middle|\frac{1}{1-x-x^2},x\right).$$

Then, we have

$$D_3=D_1+D_2=\left(\frac{3-x}{1-x-x^2}\middle|\frac{2-x^2}{x^3-2x+1},x\right).$$

Namely,

$$\begin{pmatrix}3&0&0&0&0&0&\cdots\\2&2&0&0&0&0&\cdots\\5&4&2&0&0&0&\cdots\\7&7&4&2&0&0&\cdots\\12&12&7&4&2&0&\cdots\\19&20&12&7&4&2&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}=\begin{pmatrix}1&0&0&0&0&0&\cdots\\1&1&0&0&0&0&\cdots\\2&3&1&0&0&0&\cdots\\3&5&3&1&0&0&\cdots\\5&9&5&3&1&0&\cdots\\8&15&9&5&3&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

$$+\begin{pmatrix}2&0&0&0&0&0&\cdots\\1&1&0&0&0&0&\cdots\\3&1&1&0&0&0&\cdots\\4&2&1&1&0&0&\cdots\\7&3&2&1&1&0&\cdots\\11&5&3&2&1&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

Also, the generating functions of the A, Z and ω-sequences are

$$A_1(x)=1 \tag{3.5}$$

$$Z_1(x)=\frac{2x^2-3x-1}{x^2-x-1} \tag{3.6}$$

$$\omega_1(x) = \frac{2x^2 - 2x - 1}{x^2 - x - 1}, \tag{3.7}$$

and

$$A_2(x) = 1 \tag{3.8}$$

$$Z_2(x) = \frac{1}{2}(1 + x) \tag{3.9}$$

$$\omega_2(x) = \frac{1}{2}(1 + 5x). \tag{3.10}$$

Now, we find the A_3, Z_3 and ω_3-sequences of D_3. If we use (3.4), we obtain the generating function of the A_3-sequence as follows

$$A_3(x) = 1. \tag{3.11}$$

From here the first few elements of the A_3-sequence are

$$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots$$

To find the Z_3-sequence, using (3.2), (3.6), (3.9) and (3.11), we get

$$\begin{aligned} Z_3(x) = {} & \frac{2}{3\left(\frac{1+x-x^2}{1-2x+x^3} + \frac{1}{1-x-x^2}\right)} \left(\frac{2x^2 - 3x - 1}{x^2 - x - 1} \frac{1 + x - x^2}{1 - 2x + x^3} + (1 + x) \frac{1}{1 - x - x^2} \right) \\ & + \frac{\left(\frac{1+x-x^2}{1-2x+x^3} - \frac{1}{1-x-x^2}\right)}{3\left(\frac{1+x-x^2}{1-2x+x^3} + \frac{1}{1-x-x^2}\right)} \\ & = \frac{7x^2 - 8x - 4}{3(x^2 - 2)}. \end{aligned}$$

The first terms of the Z_3-sequence are

$$\frac{2}{3}, \frac{4}{3}, -\frac{5}{6}, \frac{2}{3}, -\frac{5}{12}, \frac{1}{3}, -\frac{5}{24}, \frac{1}{6}, -\frac{5}{48}, \frac{1}{12}, \ldots.$$

Using (3.3), (3.7), (3.10) and (3.11), the generating function of the ω_3-sequence is as follows

$$\begin{aligned} \omega_3(x) &= \frac{2}{3} + \frac{1}{3} \frac{5x\left((3 - 2x)\frac{3-x}{1-x-x^2} - 9\right)\left(\frac{2x^2-2x-1}{x^2-x-1} - 1\right)}{5x\left((1 - x)\frac{1}{1-x-x^2} - 1\right) + \left(\frac{2x^2-2x-1}{x^2-x-1} - 1\right)\left(\frac{(2-x)^2}{1-x-x^2} - 4\right)} \\ &= \frac{13x^2 - 11x - 4}{3(x^2 - 2)} \end{aligned}$$

and the first few elements of the ω_3-sequence are

$$\frac{2}{3}, \frac{11}{6}, -\frac{11}{6}, \frac{11}{12}, -\frac{11}{12}, \frac{11}{24}, -\frac{11}{24}, \frac{11}{48}, -\frac{11}{48}, \ldots.$$

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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