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Some graph parameters on the strong product of monogenic semigroup graphs

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Abstract

In Das et al. (2013), it has been defined a new algebraic graph on monogenic semigroups. Our main scope in this study, is to extend this study over the special algebraic graphs to the strong product. In detail, we will determinate some important graph parameters (diameter, girth, radius, maximum degree, minimum degree, chromatic number, clique number and domination number) for the strong product of any two monogenic semigroup graphs.

Keywords: *Monogenic semigroup graph, strong product, graph parameters.*

Monojenik yarıgrup graflarının güçlü çarpımlarının bazı graf parametreleri

Özet

Das ve diğ. (2013) çalışmasında monojenik yarıgruplar üzerinde yeni bir cebirsel graf tanımlanmıştır. Bu çalışmada ana odaklanma noktamız, bu çalışmayı verilen özel cebirsel grafların güçlü çarpımına genişletmektir. Detaylandıracak olursak, herhangi iki monojenik yarıgrup graflarının güçlü çarpımları için bazı önemli graf parametrelerini (çap, çevrim, yarıçap, maksimum derece, minimum derece, renklendirme sayısı, klik sayısı ve baskınlık sayısı) hesaplayacağız.

Anahtar kelimeler: *Monojenik yarıgrup grafları, güçlü çarpım, graf parametreleri.*

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1. Introduction

A very huge number of studies about zero-divisor graphs have been stranded in the literature. First study of zero-divisor graphs started with commutative rings in the paper [2]. After that, many authors also studied about that graph over rings and semigroups (see, for instances, [3-7]).

The graph $\Gamma(S_M)$ on monogenic semigroups S_M (with zero) having elements $\{0, x, x^2, ..., x^n\}$. The vertices of this graph are the non-zero elements $x, x^2, ..., x^n$ and any two different vertices x^i and x^j are adjacent iff $i + j \geq n+1$ (for $1 \leq i, j \leq n$) [1].

In this paper, we consider the strong product of monogenic semigroup graphs and we obtained some results for the diameter, girth, radius, maximum degrees, minimum degrees, clique number, chromatic number and domination number.

It is known that studying the *product* of graphs is also an important subject (for instance, [8-14]) since there are so many applications in sciences.
Firstly, we will give some information about tensor and lexicographic product of monogenic semigroup graphs.

The lexicographic product of monogenic semigroup graphs as follows [15];
Let us take any two vertices (x^i, x^j) and (x^a, x^b) which are connected if and only if

$$\begin{cases} x^i x^a \in E(\Gamma(S_m^1)) \Leftrightarrow x^i x^a = 0 \Leftrightarrow i + a \geq n+1 \\ or \\ x^i = x^a \text{ and } x^j x^b \in (\Gamma(S_m^2)) \Leftrightarrow x^i = x^a \text{ and } j + b \geq m+1. \end{cases} \tag{1}$$

The tensor product of monogenic semigroup graphs as follows [16];
Let us take any two vertices (x^i, x^j) and (x^a, x^b) which are connected if and only if

$$\begin{cases} x^i x^a \in E(\Gamma(S_m^1)) \Leftrightarrow x^i x^a = 0 \Leftrightarrow i + a \geq n+1 \\ and \\ x^j x^b \in E(\Gamma(S_m^2)) \Leftrightarrow x^j x^b = 0 \Leftrightarrow j + b \geq m+1. \end{cases} \tag{2}$$

In previous studies [15,16] some properties like diameter, girth, maximum and minimum degree etc. of monogenic semigroup graphs have been established.

Now, we will establish these properties for strong product of monogenic semigroup graphs.

With this idea, it is defined the *strong product* $G_1 \boxtimes G_2$ of any two simple graphs G_1 and G_2 which has the vertex set $V(G_1) \times V(G_2)$ such that any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are connected to by an edge: ($u_1 = v_1$ and $u_2 v_2 \in E(G_2)$) or ($u_2 = v_2$ and $u_1 v_1 \in E(G_1)$) or ($u_1 v_1 \in E(G_1)$ and $u_2 v_2 \in E(G_2)$) (see, for instance,

[9,13,14]). In here, we will replace G_1 by $\Gamma(S_M^1)$ and G_2 by $\Gamma(S_M^2)$ we have rules for monogenic semigroup graphs as follows:
$\Gamma(S_M^1)\boxtimes\Gamma(S_M^2)$ has vertex set $\Gamma(S_M^1)\times\Gamma(S_M^2)$ and let us take any two vertices (x^i, x^j) and (x^a, x^b) which are connected if and only if:

$$\begin{cases} x^a = x^i \text{ and } x^b x^j = 0 \Leftrightarrow a = i \text{ and } b + j \geq m+1 \\ or \\ x^a x^i = 0 \text{ and } x^b = x^j \Leftrightarrow a+i \geq n+1 \text{ and } b = j \\ or \\ x^a x^i = 0 \text{ and } x^b x^j = 0 \Leftrightarrow a+i \geq n+1 \text{ and } b+j \geq m+1. \end{cases} \tag{3}$$

In this paper, by considering $\Gamma(S_M^1)\boxtimes\Gamma(S_M^2)$, we will give some good results for the diameter, radius, girth, maximum degree, minimum degree, domination number, chromatic number, clique number.

2. Main Results

First result is about diameter which is well-known graph parameter.

The *distance* of a simple, connected graph G is the length of the shortest path between two vertices a and b and denoted by $d_G(a,b)$. Also, the *diameter* of a simple, connected graph G is equal to $diam(G) = \max\{d_G(a,b) : a,b \in V(G)\}$ [17].

2.1. Theorem
$diam(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2)) = 2$.

Proof.
The vertices (x^i, x^j) and $(x^a, x^b) \in \Gamma(S_M^1)\boxtimes\Gamma(S_M^2)$ they are not adjacent. Also (x^n, x^m) is adjacent to both of them and the case $a = 1$ and $j = b$ does not provide this condition.
Then; there exist an adjacency $\left(x^a, x^b\right) \sim (x^n, x^m) \sim (x^i, x^j)$.
So diameter of $\Gamma(S_M^1)\boxtimes\Gamma(S_M^2)$ is 2.

The *girth* of a simple, connected graph G is the length of the shortest cycle in the graph. If the graph G doesn't contain any cycle, then the girth is taken as infinite [17].

2.2. Theorem
$girth(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2)) = 3.$

Proof.
By considering strong product rule, we easily see that the equalities

- $n = n$ and $m+2 > m$ imply $\left(x^n, x^m\right) \sim \left(x^n, x^2\right)$,
- $n = n$ and $m+m-1 > m$ imply $\left(x^n, x^m\right) \sim \left(x^n, x^{m-1}\right)$,

- $n=n$ and $2+m-1>m$ imply $\left(x^n,x^2\right)\sim\left(x^n,x^{m-1}\right)$.

Then we can say that;

$$\left(x^n,x^m\right)\sim\left(x^n,x^2\right)\sim\left(x^n,x^{m-1}\right)\sim\left(x^n,x^m\right)$$

so as desired.

The *eccentricity* of a vertex u, shown $e(u)$, in a connected graph G is the maximum distance between u and any other vertex v of G. It is clear that diameter of a graph is equal to the maximum eccentricity of G. Also the minimum eccentricity is equal to the *radius* of G and denoted by $rad(G)=\min_{u}\left\{\max_{v}\left\{d_G(u,v)\right\}\right\}$ [17,18].

2.3. Theorem
$rad(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2))=1.$

Proof.
We know that the vertex $\left(x^n,x^m\right)$ is adjacent to all other vertices for any vertex $\left(x^i,x^j\right)$, $1\le i\le n,\ 1\le j\le m$ is adjacent to $\left(x^n,x^m\right)$ so the distance is equal to 1.

The degree of a vertex v of G $\left(\deg_G(v)\right)$ is the number of vertices adjacent to v. Among all degrees, the maximum $\Delta\left(G\right)$ (or the minimum $\delta\left(G\right)$) degrees of G is the number of the largest (or smallest) degree in G [19].

2.4. Theorem
$\Delta(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2))=n.m-1$ and $\delta(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2))=3.$

Proof.
The vertex set of $V(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2))$ has $n.m$ vertices. Let us take the vertices (x^n,x^m) because this vertex is adjacent to all other vertices. That's why the maximum degree $\left(\Delta\right)$ of the graph is equal to $n.m-1$.

Now, let us take the vertex $\left(x,x\right)$ because this vertex is just adjacent to vertices $\left(x^n,x\right),\left(x,x^m\right)$ and $\left(x^n,x^m\right)$. then it is clear that the minimum degree $\left(\delta\right)$ of the graph is equal to 3.

A subset A of the vertex set $V\left(G\right)$ of a graph is collect the domination set if every vertex $V\left(G\right)-A$ is joined to at least one vertex of A by an edge. The *domination number* $\gamma\left(G\right)$ is the number of vertices in the smallest dominating set for G [19].

2.5. Theorem
$\gamma\left(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2)\right)=1$.

Proof.
Let us take the vertex $\left(x^n,x^m\right)$ of the graph $\Gamma(S_M^1)\boxtimes\Gamma(S_M^2)$. This vertex is adjacent to all other vertices hence. We choose this vertex as a dominating set so the domination number of graph is equal to 1.

The coloring of a graph G is defined as an assignment of colors to the vertices of G. One color to each vertex so that the adjacent vertices are assigned different colors. If in colors are used then it is called *n-coloring*. The minimum number of n is called the *chromatic number* and denoted by $\chi(G)$ [19].

2.6. Theorem
$\chi(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2))=\left(1+\lceil\frac{n-1}{2}\rceil\right).\left(1+\lceil\frac{m-1}{2}\rceil\right)$

Proof.
Let us take the vertex $\left(x^n,x^m\right)$ and the color a_1^1 was used for this vertex. This vertex is adjacent to all other vertices so a_1^1 cannot be used for other vertices. After that let choose the vertex$\left(x^n,x^{m-1}\right)$, then it is obvious that this vertex is not adjacent to $\left(x^n,x\right)$ so we can use same color a_1^2 for these vertices.

Similary let take the vertex $\left(x^n,x^{m-2}\right)$ is adjacent to all vertices except the vertices $\left(x^n,x\right)$ and $\left(x^n,x^2\right)$ so we can use same color a_1^3 for these vertices.
If we apply the same steps to other vertices $\left(x^n,x^i\right)$, $1\le i\le m$. We obtained that $1+\lceil\frac{m-1}{2}\rceil$ different colors.

Now, let choose the vertex $\left(x^{n-1},x^m\right)$. This vertex is not adjacent to $(x,x),\left(x,x^2\right),...,\left(x,x^{\lceil\frac{m}{2}\rceil}\right),\left(x,x^{\lceil\frac{m}{2}\rceil+1}\right),...,\left(x,x^m\right)$. Also, there exist adjacency between the vertices $\left(x,x^{\lceil\frac{m}{2}\rceil+1}\right),\left(x,x^{\lceil\frac{m}{2}\rceil+2}\right),...,\left(x,x^m\right)$.

Because of this the color of a_2^1 which is used for the vertex $\left(x^{n-1},x^m\right)$ can be used for the vertices$(x,x),\left(x,x^2\right),...,\left(x,x^{\lceil\frac{m}{2}\rceil}\right)$. Then, consider the vertex $\left(x^{n-1},x^{m-1}\right)$ which is not adjacent to the vertices $(x,x),\left(x,x^2\right),...,\left(x,x^m\right)$ and $\left(x^{n-1},x\right)$. The color of which is used for $\left(x^{n-1},x^{m-1}\right)$ also can be used for the vertices $\left(x,x^{\lceil\frac{m}{2}\rceil+1}\right)$ and $\left(x^{n-1},x\right)$.

Let choose the final vertex $\left(x^{n-1},x^{m-2}\right)$ which is not adjacent to the vertices $(x,x),\left(x,x^2\right),...,\left(x,x^m\right)$, $\left(x^{n-1},x\right)$ and $\left(x^{n-1},x^2\right)$. As before, the color a_2^3 used for $\left(x^{n-1},x^{m-2}\right)$, $\left(x,x^{\lceil\frac{m}{2}\rceil+2}\right)$ and $\left(x^{n-1},x^2\right)$.
Finally, we can say that $\left(1+\lceil\frac{m-1}{2}\rceil\right)$ different colors are necessary for the coloring of vertices in the set $\left\{\left(x^{n-1},x^j\right),\left(x,x^i\right):1\leq j\leq m,\lfloor\frac{m}{2}\rfloor+1\leq i\leq m\right\}$.

Last part of the proof, let take the vertex $\left(x^{n-2},x^m\right)$ which is not adjacent to the vertices $(x,x),\left(x,x^2\right),...,\left(x,x^m\right),\left(x^2,x\right),\left(x^2,x^2\right),...,\left(x^2,x^m\right)$. Also these vertices $\left(x^2,x\right),\left(x^2,x^2\right),...,\left(x^2,x^m\right)$ are not adjacent to $\left(x^{n-2},x^m\right)$. So we can use the same color a_3^1 for these vertices and $\left(x^{n-2},x^m\right)$.

By applying the same process, let choose the vertex $\left(x^{n-2},x^{m-1}\right)$. The color a_3^2 is used only for $\left(x^2,x^{\lceil\frac{m}{2}\rceil+1}\right)$ and $\left(x^{n-2},x^{m-1}\right)$.

Also the vertex $\left(x^{n-2},x^{m-2}\right)$ is not adjacent to vertices $\left(x^2,x\right),\left(x^2,x^2\right),...,\left(x^2,x^m\right),\left(x^{n-2},x\right)$ and $\left(x^{n-2},x^2\right)$ so the color a_3^3 can be used only for the vertices $\left(x^2,x^{\lceil\frac{m}{2}\rceil+2}\right)$ and $\left(x^{n-2},x^2\right)$.

Now, we obtained $\left(1+\lceil\frac{m-1}{2}\rceil\right)$ different colors for the vertices $\left\{\left(x^{n-2},x^j\right),\left(x^2,x^i\right):1\leq j\leq m,\lceil\frac{m}{2}\rceil+1\leq i\leq m\right\}$.

If we apply the same process to all vertices we need to $\left(1+\lceil\frac{n-1}{2}\rceil\right)$ steps. Then we obtained that

$$\chi\left(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2)\right)=\left(1+\lceil\tfrac{n-1}{2}\rceil\right).\left(1+\lceil\tfrac{m-1}{2}\rceil\right).$$

All complete subgraphs of a graph are called "*clique*". The *clique number* of a graph is equal to the maximum vertex number of a clique. The clique number is denoted by $\omega(G)$ [19].

2.7. Theorem
$\omega(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2))=\left(1+\lceil\frac{n-1}{2}\rceil\right).\left(1+\lceil\frac{m-1}{2}\rceil\right)$

Proof.
In the proof, we must .first check whether the subgraph is complete or not. Now let us consider definition of the graph $\Gamma(S_M^1)\boxtimes\Gamma(S_M^2)$. Then will have a maximal complete subgraph with the vertex set:

$$V(A)=\left\{\begin{array}{l}\left(x^{\lceil\frac{n}{2}\rceil},x^{\lceil\frac{m}{2}\rceil}\right),\left(x^{\lceil\frac{n}{2}\rceil},x^{\lceil\frac{m}{2}\rceil+1}\right),\ldots,\left(x^{\lceil\frac{n}{2}\rceil},x^{m}\right),\\ \left(x^{\lceil\frac{n}{2}\rceil+1},x^{\lceil\frac{m}{2}\rceil}\right),\left(x^{\lceil\frac{n}{2}\rceil+1},x^{\lceil\frac{m}{2}\rceil+1}\right),\ldots,\left(x^{\lceil\frac{n}{2}\rceil},x^{m}\right),\\ \left(x^{n},x^{\lceil\frac{m}{2}\rceil}\right),\left(x^{n},x^{\lceil\frac{m}{2}\rceil+1}\right),\ldots,\left(x^{n},x^{m}\right).\end{array}\right\}$$

So $\omega(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2))=\left(1+\lceil\frac{n-1}{2}\rceil\right).\left(1+\lceil\frac{m-1}{2}\rceil\right)$ as desired.

2.8. Remark

By Theorems 2.6 and 2.7,

$$\chi(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2))=\omega(\Gamma(S_M^1)\boxtimes\Gamma(S_M^2))=\left(1+\lceil\frac{n-1}{2}\rceil\right).\left(1+\lceil\frac{m-1}{2}\rceil\right)$$

which implies that the strong product preserves the perfectness property [19] for the important graphs $\Gamma(S_M^1)$ and $\Gamma(S_M^2)$.

2.9. Example

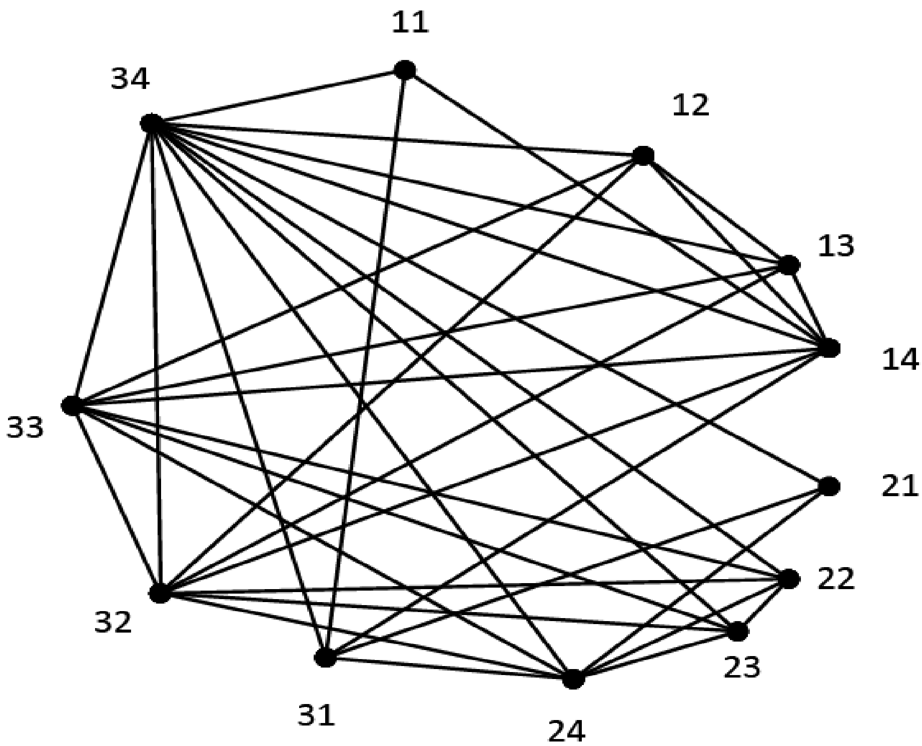

Figure 1. The graph $\Gamma(S_{M_3}^1)\boxtimes\Gamma(S_{M_4}^2)$. Here for $1\le a\le 3$ and $1\le b\le 4$ each label ab corresponds to the vertices $\left(x^a,x^b\right)$.

Let us consider the semigroups

$S^1_{M_3} = \{x, x^2, x^3\}$ and $S^2_{M_4} = \{x, x^2, x^3, x^4\}$

and then let us give our attention to the graph $\Gamma(S^1_{M_3}) \boxtimes \Gamma(S^2_{M_4})$ as drawn in Fig. 1. Depend on the results presented in this study, we can state the following results:

i. $diam(\Gamma(S^1_{M_3}) \boxtimes \Gamma(S^2_{M_4})) = 2$ (by Theorem 2.1).
ii. $girth(\Gamma(S^1_{M_3}) \boxtimes \Gamma(S^2_{M_4})) = 3$ (by Theorem 2.2).
iii. $rad(\Gamma(S^1_{M_3}) \boxtimes \Gamma(S^2_{M_4})) = 1$ (by Theorem 2.3).
iv. $\Delta(\Gamma(S^1_{M_3}) \boxtimes \Gamma(S^2_{M_4})) = 11$ and $\delta(\Gamma(S^1_{M_3}) \boxtimes \Gamma(S^2_{M_4})) = 3$ (by Theorem 2.4).
v. $\gamma\left(\Gamma(S^1_{M_3}) \boxtimes \Gamma(S^2_{M_4})\right) = 1$ (by Theorem 2.5).
vi. $\chi(\Gamma(S^1_{M_3}) \boxtimes \Gamma(S^2_{M_4})) = 6$ (by Theorem 2.6).
vii. $\omega(\Gamma(S^1_{M}) \boxtimes \Gamma(S^2_{M})) = 6$ (by Theorem 2.7).

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