

q -Riordan array for q -Pascal matrix and its inverse matrix

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Received: 18.06.2015

Accepted/Published Online: 16.12.2015

Final Version: 21.10.2016

Abstract: In this paper, we prove the q -analogue of the fundamental theorem of Riordan arrays. In particular, by defining two new binary operations $*_q$ and $*_{1/q}$, we obtain a q -analogue of the Riordan representation of the q -Pascal matrix. In addition, by aid of the q -Lagrange expansion formula we get q -Riordan representation for its inverse matrix.

Key words: Riordan representation, Pascal matrices, q -calculus

1. Introduction

Infinite triangular matrices are usually used in combinatorics and matrix theory. Particularly, Pascal matrices play an important role in the study of matrix theory. A Pascal matrix is a special matrix with entries formed by binomial coefficients. Gaussian numbers, also called q -binomial coefficients, are the q -analogues of usual binomial coefficients. q -calculus, which mathematicians say is calculus without limits, is studied in many different of branches of mathematics. In particular, Ernst presented the tools and methods for q -calculus [9]. Analogous to Pascal matrices and considering q -calculus, q -binomial coefficients are used to obtain a new matrix called the q -Pascal matrix. Special combinatorial properties of q -Pascal matrices were studied by Ernst in [8]. Srivastava also obtained some useful formulas for q -generating functions [15, 16]. Using these q -functions and the concept of the Riordan array, a new area can be constructed. Riordan arrays form an important study area in matrix theory. Riordan arrays are infinite, lower triangular matrices defined by two generating functions. Initially, Shapiro [13] formed a group called the Riordan group. Sprugnoli [14] studied combinatorial sums and identities of Riordan arrays. Barry characterized the special triangles using Riordan arrays and obtained the properties of generalized Pascal matrices, defined by Riordan arrays. Barry also got the inverse of Pascal-like matrices [1–3].

Tuglu et al. [17] established a relationship between Riordan arrays and Fibonomial coefficients. Associating Riordan arrays to q -binomial coefficients, a new direction of study is initiated. Cheon et al. [7] proved that a q -Riordan matrix can be represented by using Eulerian generating functions with usual binary operations.

In this direction of study, usual operations and definitions fail to satisfy several properties. Therefore, we define two new binary operations denoted by $*_q$ and $*_{1/q}$. By using these operations, m th powers of any function are rewritten. By aid of these symbolic powers, we define q -matrices that can be represented by $(g, f)_q$

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2010 AMS Mathematics Subject Classification: 15A15; 15B48; 05A30.

or $(g, f)_{1/q}$. Then we obtain the q -analogue of the fundamental theorem of Riordan arrays. As a result of this theorem we get a representation for the q -Pascal matrix. Finally, with the use of the q -Lagrange inversion formula and this representation, we get the q -analogue of the Riordan representation of its inverse matrix.

2. Preliminaries

Let n and k be integers with $n \geq k \geq 1$. The q -binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{(1 - q^n)(1 - q^{n-1}) \dots (1 - q^{n-k+1})}{(1 - q^k)(1 - q^{k-1}) \dots (1 - q)}, \tag{2.1}$$

with $\binom{n}{0}_q = 1$ and $\binom{n}{k}_q = 0$ for $n < k$ [5]. The q -binomial coefficient is reduced to the binomial coefficient as $q \rightarrow 1^-$, that is,

$$\binom{n}{k}_q \longrightarrow \binom{n}{k}.$$

Let $0 \leq i, j \leq n - 1$. The $n \times n$ Pascal matrix $P_n = (p_{ij})$ is defined by

$$p_{ij} = \begin{cases} \binom{i}{j} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

In the literature, several types of Pascal matrices are defined and studied [4, 11, 12, 18, 19]. The q -analogue of the Pascal matrix is denoted by $\mathcal{P} = (\mathfrak{p}_{ij})$ and is defined by

$$\mathfrak{p}_{ij} = \begin{cases} q^{\binom{j+1}{2}} \binom{i}{j}_q & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

It is well known that q -binomial coefficients satisfy the following inversion formula:

$$\sum_{j \geq k} \binom{n}{j}_q \binom{j}{k}_q q^{\binom{j-k}{2}} (-1)^{j-k} = \delta_{n,k}, \tag{2.4}$$

where $\delta_{n,k} = 1$ if $n = k$ and zero otherwise. Simple modifications in (2.4) yields

$$\sum_{j \geq k} \binom{n}{j}_q q^{\binom{j+1}{2}} \binom{j}{k}_q q^{\binom{j-k}{2}} q^{-\binom{j+1}{2}} (-1)^{j-k} = \delta_{n,k}.$$

This implies that the inverse of $\mathcal{P} = (\mathfrak{p}_{ij})$ is $\mathcal{P}^{-1} = (\mathfrak{p}'_{ij})$ where

$$\mathfrak{p}'_{ij} = \begin{cases} (-1)^{i-j} q^{\binom{j+1}{2} - i(j+1)} \binom{i}{j}_q & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Next, we briefly review the basic facts about the Riordan group.

Definition 2.1 ([13]) Let g and f be two formal power series of the form

$$g(x) = g_0 + g_1x + g_2x^2 + \dots \tag{2.6}$$

and

$$f(x) = f_1x + f_2x^2 + f_3x^3 + \dots \tag{2.7}$$

with $g_0 \neq 0$ and $f_1 \neq 0$. Let (g, f) be the infinite lower-triangular matrix with the j th column formed by the coefficients of the power series

$$g(x)f(x)^j, \quad j = 0, 1, 2, \dots \tag{2.8}$$

Let \mathcal{R} be the set of all infinite lower-triangular matrices defined by (2.8) and let (g, f) and $(u, v) \in \mathcal{R}$. Then \mathcal{R} is a group under the operation

$$(g, f) * (u, v) := (g(u \circ f), v \circ f).$$

In particular, \mathcal{R} is called the Riordan group and any element (g, f) of \mathcal{R} is called a Riordan pair. The identity element of \mathcal{R} is $I = (1, x)$ and the inverse of any (g, f) is

$$(g, f)^{-1} = \left(\frac{1}{g \circ \bar{f}}, \bar{f} \right), \tag{2.9}$$

where \bar{f} is the compositional inverse of f .

Let

$$f(z) = \sum_{n \geq 0} c_n \frac{z^n}{(1-z)_q^n},$$

where $(1-z)_q^n := (1-z)(1-qz) \cdots (1-q^{n-1}z)$, and c_n is independent of z . Carlitz showed that

$$f(z) = f(0) + \sum_{n \geq 1} \frac{1}{(1-q)_q^n} \frac{z^n}{(1-z)_q^n} \Delta_0^{n-1} \left\{ \Delta f(z) (1-z)_q^n \right\} \tag{2.10}$$

[6]. This formula is called the q -analogue of a special case of Lagrange expansion.

For a given $\theta(z) = \sum_{n \geq 0} \theta_n z^n$, roofing and starring operators are defined respectively by

$$\hat{\theta}(z) = \sum_{n \geq 0} \theta_n q^{-\binom{n}{2}} z^n$$

and

$$\theta^*(z) = \theta(z)\theta(qz)\theta(q^2z) \cdots$$

Let $f(z) = \frac{z}{r(z)}$ be the inverse of $F(z) = \frac{z}{R(z)}$. Using the above operators, Garsia [10] defined the q -analogue of functional composition as

$$\Phi(\bar{F}) = \sum_{n \geq 0} \Phi_n F(z) F(qz) \cdots F(q^{n-1}z) = \frac{(\Phi(z)^* r(z))^\vee}{*\check{r}(z)} \tag{2.11}$$

and

$$\Phi(\underline{f}) = \sum_{n \geq 0} \Phi_n f(z) f(z/q) \cdots f(z/q^{n-1}) = \frac{(\Phi(z) R^*(z))^\wedge}{\hat{R}^*(z)}. \tag{2.12}$$

3. The q -analogue of Riordan representation

In this section, we study the q -analogue of the fundamental theorem of Riordan arrays (FTRA). For this purpose, we first define two binary operations, $*_q$ and $*_{1/q}$. Then we use the $*_q$ operation to give a theorem that we call the q -analogue of the FTRA.

Definition 3.1 Let $\mathcal{F}_q(n)$ be the set of generating functions of the form

$$a_n(q)x^n + a_{n+1}(q)x^{n+1} + a_{n+2}(q)x^{n+2} + \dots, \tag{3.1}$$

with $a_n(q) \neq 0$. Let $g(x) \in \mathcal{F}_q(0)$ and $f(x) \in \mathcal{F}_q(1)$. We define two binary operations $*_q, *_{1/q} : \mathcal{F}_q(0) \times \mathcal{F}_q(1) \rightarrow \mathcal{F}_q(1)$ by

$$g(x) *_q f(x) = g(x) \cdot f(qx)$$

and

$$g(x) *_{1/q} f(x) = g(x) \cdot f(x/q).$$

In addition, we give the m th power of f by

$$f^{[\bar{m}]}(x) = f(x) *_q f^{[\bar{m-1}]}(x), \quad (m \geq 1),$$

where $f^{[0]}(x) = 1$. Similarly,

$$f^{-[m]}(x) = f(x) *_{1/q} f^{-[m-1]}(x), \quad (m \geq 1),$$

where $f(x)^{[0]} = 1$.

Let $g(x) \in \mathcal{F}_q(0)$ and $f(x) \in \mathcal{F}_q(1)$. We denote by $(g, f)_q$ the infinite lower-triangular matrix whose j th column is formed by the coefficients of the power series expansion of

$$g(x) *_q f^{[j]}(x), \quad j = 0, 1, 2, \dots \tag{3.2}$$

Using the binary operation $*_q$, we obtain a representation that is the analogue of the Riordan representation. We call it the q -analogue of the Riordan representation and denote it by $(g, f)_q$. Similarly, using $*_{1/q}$, we get $(g, f)_{1/q}$.

Theorem 3.2 (The q -analogue of FTRA) Let $g(x) \in \mathcal{F}_q(0)$ and $f(x) \in \mathcal{F}_q(1)$. Let $A(x) = \sum_{k \geq 0} a_k(q) x^k$ and $B(x) = \sum_{k \geq 0} b_k(q) x^k$. Then we have

$$(g(x), f(x))_q \cdot \begin{bmatrix} a_0(q) \\ a_1(q) \\ a_2(q) \\ a_3(q) \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0(q) \\ b_1(q) \\ b_2(q) \\ b_3(q) \\ \vdots \end{bmatrix}, \tag{3.3}$$

if and only if the following equation holds:

$$g(x) *_q A(\bar{f}(x)) = B(x). \tag{3.4}$$

Proof Let $g(x) \in \mathcal{F}_q(0)$ and $f(x) \in \mathcal{F}_q(1)$, that is,

$$g(x) = \sum_{k=0}^{\infty} g_k(q)x^k, \quad f(x) = \sum_{k=1}^{\infty} f_k(q)x^k.$$

For simplicity, we let $a_k(q) = a_k$, $b_k(q) = b_k$, $f_k(q) = f_k$, $g_k(q) = g_k$. Then the left-hand side of (3.3) becomes

$$\begin{bmatrix} | & & & & \dots \\ | & & & & \\ g & g *_q f^{[1]} & g *_q f^{[2]} & & \\ | & | & | & | & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix}.$$

This can be rewritten as

$$\begin{bmatrix} g_0 & 0 & 0 & 0 & \dots \\ g_1 & g_0 f_1 q & 0 & 0 & \\ g_2 & g_0 f_2 q^2 + g_1 f_1 q & g_0 f_1^2 q^3 & 0 & \\ g_3 & g_0 f_3 q^3 + g_1 f_2 q^2 + g_2 f_1 q & g_0 f_1 f_2 q^5 + g_0 f_1 f_2 q^4 + g_1 f_1^2 q^3 & g_0 f_1^3 q^6 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix}.$$

Carrying out the multiplication, we obtain the following column vector:

$$\begin{bmatrix} g_0 a_0 \\ g_1 a_0 + g_0 f_1 q a_1 \\ g_2 a_0 + (g_0 f_2 q^2 + g_1 f_1 q) a_1 + g_0 f_1^2 q^3 a_2 \\ g_3 a_0 + (g_0 f_3 q^3 + g_1 f_2 q^2 + g_2 f_1 q) a_1 + (g_0 f_1 f_2 q^5 + g_0 f_1 f_2 q^4 + g_1 f_1^2 q^3) a_2 + g_0 f_1^3 q^6 a_3 \\ \vdots \end{bmatrix}$$

Let L be the product of the above column vector and $[1, x, x^2, \dots]$, i.e.

$$\begin{aligned} L &= g_0 a_0 + [g_1 a_0 + g_0 f_1 q a_1] x + [g_2 a_0 + (g_0 f_2 q^2 + g_1 f_1 q) a_1 + g_0 f_1^2 q^3 a_2] x^2 \\ &\quad + [g_3 a_0 + (g_0 f_3 q^3 + g_1 f_2 q^2 + g_2 f_1 q) a_1 + (g_0 f_1 f_2 q^5 + g_0 f_1 f_2 q^4 + g_1 f_1^2 q^3) a_2 + g_0 f_1^3 q^6 a_3] x^3 \\ &\quad + \dots \end{aligned}$$

Then L can be rewritten as

$$\begin{aligned} L &= a_0 \cdot [g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \dots] \\ &\quad + a_1 \cdot [g_0 f_1 q x + (g_0 f_2 q^2 + g_1 f_1 q) x^2 + (g_0 f_3 q^3 + g_1 f_2 q^2 + g_2 f_1 q) x^3 + \dots] \\ &\quad + a_2 \cdot [g_0 f_1^2 q^3 x^2 + (g_0 f_1 f_2 q^5 + g_0 f_1 f_2 q^4 + g_1 f_1^2 q^3) x^3] + \dots \\ &= a_0 \cdot g(x) + a_1 \cdot [g(x) *_q f(x)] + a_2 \cdot [g(x) *_q f^{[2]}(x)] + a_3 \cdot [g(x) *_q f^{[3]}(x)] + \dots \\ &= g(x) *_q [a_0 + a_1 \cdot f^{[1]}(x) + a_2 \cdot f^{[2]}(x) + a_3 \cdot f^{[3]}(x) + \dots] \\ &= g(x) *_q A(\bar{f}(x)). \end{aligned}$$

We observe that $B(x)$ is a product of the right-hand side of (3.3) and $[1, x, x^2, \dots]$. Therefore, $L = B(x)$ if and only if (3.4) holds, as claimed. \square

Corollary 3.3 *Let \mathcal{P} be the q -Pascal matrix given in (2.3). Then the q -analogue of the Riordan representation of \mathcal{P} is*

$$\mathcal{P} = \left(\frac{1}{(1-x)_q}, \frac{x}{(1-x)_q} \right)_q. \tag{3.5}$$

Proof The matrix \mathcal{P} can be written as follows.

$$\mathcal{P} = \begin{bmatrix} \binom{0}{0}_q & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \binom{1}{0}_q & q\binom{1}{1}_q & 0 & 0 & 0 & 0 & 0 & \dots \\ \binom{2}{0}_q & q\binom{2}{1}_q & q^3\binom{2}{2}_q & 0 & 0 & 0 & 0 & \dots \\ \binom{3}{0}_q & q\binom{3}{1}_q & q^3\binom{3}{2}_q & q^6\binom{3}{3}_q & 0 & 0 & 0 & \dots \\ \binom{4}{0}_q & q\binom{4}{1}_q & q^3\binom{4}{2}_q & q^6\binom{4}{3}_q & q^{10}\binom{4}{4}_q & 0 & 0 & \dots \\ \vdots & \dots \end{bmatrix}$$

The entry (i,j) of \mathcal{P} is given by

$$p_{ij} = [x^i] \sum_{i=0}^{\infty} q^{\binom{j+1}{2}} \binom{i}{j}_q x^i.$$

Therefore, the j th column is formed by the coefficients of the power series

$$\sum_{i=0}^{\infty} q^{\binom{j+1}{2}} \binom{i}{j}_q x^i.$$

Consider two q -functions

$$g(x) = \frac{1}{(1-x)_q} \quad \text{and} \quad f(x) = \frac{x}{(1-x)_q}.$$

Then we have

$$\begin{aligned} f^{[\bar{j}]}(x) &= f(x) *_q f^{[j-1]}(x) \\ &= \frac{x}{(1-x)_q} *_q \left(\frac{x}{(1-x)_q} \right)^{[j-1]} \\ &= \frac{x}{1-x} \cdot \frac{qx}{1-qx} \cdots \frac{q^{j-1}x}{1-q^{j-1}x} \\ &= \frac{q^{\binom{j}{2}} x^j}{(1-x)_q^j} \end{aligned}$$

and

$$\begin{aligned} g(x) *_q f(x)^{[\bar{j}]} &= \frac{1}{(1-x)_q} *_q \frac{q^{\binom{j}{2}} x^j}{(1-x)_q^j} \\ &= \frac{q^{\binom{j+1}{2}} x^j}{(1-x)_q^{j+1}}. \end{aligned}$$

Using the Heine binomial formula, we obtain

$$\begin{aligned} g(x) *_q f(x)^{[\bar{j}]} &= \sum_{k=0}^{\infty} q^{\binom{j+1}{2}} \binom{j+k}{j}_q x^{k+j} \\ &= \sum_{k=0}^{\infty} q^{\binom{j+1}{2}} \binom{k}{j}_q x^k. \end{aligned}$$

It shows that the generating function of the j th column of \mathcal{P} is

$$g(x) *_q f(x)^{[\bar{j}]};$$

that is,

$$\frac{q^{\binom{j+1}{2}} x^j}{(1-x)_q^{j+1}}.$$

In conclusion, the q -analogue of the Riordan representation of \mathcal{P} is

$$\left(\frac{1}{(1-x)_q}, \frac{x}{(1-x)_q} \right)_q.$$

□

Corollary 3.4 *The generic element \mathfrak{p}_{ij} of the q -Pascal matrix*

$$\mathcal{P} = \left(\frac{1}{(1-x)_q}, \frac{x}{(1-x)_q} \right)_q$$

is given by

$$\mathfrak{p}_{ij} = q^{\binom{j+1}{2}} \binom{i}{j}_q.$$

Proof

$$\begin{aligned}
 \mathfrak{p}_{ij} &= [x^i] \left\{ \frac{1}{(1-x)_q} *_{q} \left(\frac{x}{(1-x)_q} \right)^{[\bar{j}]} \right\} \\
 &= [x^i] \left\{ \frac{1}{(1-x)_q} *_{q} \left(\frac{q^{\binom{j}{2}} x^j}{(1-x)_q^j} \right) \right\} \\
 &= [x^i] \left\{ \frac{1}{(1-x)_q} \cdot \frac{q^{\binom{j+1}{2}} x^j}{(1-qx)_q^j} \right\} \\
 &= [x^i] \left\{ \frac{q^{\binom{j+1}{2}} x^j}{(1-x)_q^{j+1}} \right\} \\
 &= [x^i] \left\{ \sum_{k=0}^{\infty} q^{\binom{j+1}{2}} \binom{k}{j}_q x^k \right\} \\
 &= q^{\binom{j+1}{2}} \binom{i}{j}_q
 \end{aligned}$$

□

Lemma 3.5 Let $f(x) = \frac{x}{1-x} \in \mathcal{F}_q(1)$. Then the q -analogue of the compositional inverse of $f(x)$ is

$$h(x) = \frac{x}{1 + \frac{x}{q}}.$$

Proof Let $h(x) = \sum_{n \geq 1} h_n(q) x^n$. Then we obtain from (2.11) that

$$\begin{aligned}
 (h(\bar{f}))(x) = x &\iff \sum_{n \geq 1} h_n(q) f(x) f(qx) \cdots f(q^{n-1}x) = x \\
 &\iff \sum_{n \geq 1} h_n(q) \left(\frac{x}{1-x} \right) \left(\frac{qx}{1-qx} \right) \cdots \left(\frac{q^{n-1}x}{1-q^{n-1}x} \right) = x \\
 &\iff \sum_{n \geq 1} h_n(q) \frac{q^{\binom{n}{2}} x^n}{(1-x)_q^n} = x.
 \end{aligned}$$

In addition, let $f(x) = x$ in (2.10), and carrying out calculations, we get $h_n(q) = \frac{(-1)^{n-1}}{q^{n-1}}$. This shows that

$$h(x) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{q^{n-1}} x^n = \frac{x}{1 + \frac{x}{q}}.$$

□

Lemma 3.6 Let $g(x) = \frac{1}{1-x}$ and $h(x) = \frac{x}{1+\frac{x}{q}}$. Then

$$(g(\underline{h}))(x) = 1 + x.$$

Proof It follows from Lemma 3.5 that $h(x) = \frac{x}{1 + \frac{x}{q}}$ is the inverse of $F(x) = \frac{x}{1-x} = \frac{x}{R(x)}$. By applying (2.12) to these functions, we get

$$R^*(x) = (1 - x)(1 - qx)(1 - q^2x) \cdots$$

$$\hat{R}^*(x) = \sum_{n \geq 0} \frac{(-1)^n}{(1 - q)_q^n} x^n$$

$$g(x)R^*(x) = (1 - qx)(1 - q^2x) \cdots$$

$$(g(x)R^*(x))^\wedge = \sum_{n \geq 0} \frac{(-1)^n q^n}{(1 - q)_q^n} x^n.$$

Thus, we obtain

$$\begin{aligned} (g(\underline{h}))(x) &= \frac{(g(x)R^*(x))^\wedge}{\hat{R}^*(x)} \\ &= \frac{\sum_{n \geq 0} \frac{(-1)^n q^n}{(1 - q)_q^n} x^n}{\sum_{n \geq 0} \frac{(-1)^n}{(1 - q)_q^n} x^n} \\ &= \frac{(1 + x)(1 + qx)(1 + q^2x) \cdots}{(1 + qx)(1 + q^2x) \cdots} \\ &= (1 + x). \end{aligned}$$

This completes the proof. □

Definition 3.7 Let $f(x) \in \mathcal{F}_q(n)$. Then $\frac{1}{\bar{f}(x/q)}$ is called the q -reciprocal of $f(x)$ if $f(x) *_q \frac{1}{\bar{f}(x/q)} = 1$.

Theorem 3.8 Let \mathcal{P}^{-1} be a matrix given in (2.5). Then the q -Riordan representation of \mathcal{P}^{-1} is

$$\mathcal{P}^{-1} = \left(\frac{1}{1 + x/q}, \frac{x}{1 + x/q} \right)_{1/q}.$$

Proof It follows from (3.5) that the matrix \mathcal{P} can be represented by the pair

$$\mathcal{P} = \left(\frac{1}{(1 - x)_q}, \frac{x}{(1 - x)_q} \right)_q.$$

Therefore, we choose $g(x) = \frac{1}{(1-x)_q}$ and $f(x) = \frac{x}{(1-x)_q}$. Using the q -Lagrange expansion formula and Lemma 3.5, we obtain the q -compositional inverse of f , denoted by $\bar{f} = h$, as

$$h(x) = \frac{x}{1 + (x/q)}.$$

From Lemma 3.6, we have

$$g(\underline{h}(x)) = 1 + x$$

and the q -analogue of the reciprocal of $g(\underline{h}(x))$ as

$$\frac{1}{g(\underline{h}(x))} = \frac{1}{1 + x/q}.$$

Hence, we obtain

$$\begin{aligned} \frac{1}{g(\underline{h}(x))} *_{1/q} h^{[j]}(x) &= \frac{1}{1 + x/q} *_{1/q} \left(\frac{x}{1 + x/q} \right)^{[j]} \\ &= \left(\frac{1}{q} \right)^{\binom{j+1}{2}} x^j \prod_{i=1}^{j+1} \frac{1}{(1 + x(1/q)^i)}. \end{aligned} \tag{3.6}$$

Note that the j th column of the inverse matrix is formed by

$$\sum_{i \geq 0} (-1)^{i-j} q^{\binom{j+1}{2} - i(j+1)} \binom{i}{j}_q x^i.$$

In other words, the q -analogue of the Riordan array is formed by the coefficients of the power series expansion above for the inverse matrix. On the other hand, it is straightforward to see from the q -binomial theorem that

$$\sum_{i \geq 0} (-1)^{i-j} q^{\binom{j+1}{2} - i(j+1)} \binom{i}{j}_q x^i = \left(\frac{1}{q} \right)^{\binom{j+1}{2}} x^j \prod_{t=1}^{j+1} \frac{1}{(1 + x(1/q)^t)}. \tag{3.7}$$

It is clear that

$$\sum_{i \geq 0} (-1)^{i-j} q^{\binom{j+1}{2} - i(j+1)} \binom{i}{j}_q x^i = \frac{1}{g(\underline{h}(x))} *_{1/q} h^{[j]}(x).$$

Therefore, taking equation (2.9) into account, we obtain

$$\mathcal{P}^{-1} = \left(\frac{1}{1 + x/q}, \frac{x}{1 + x/q} \right)_{1/q}.$$

□

As $q \rightarrow 1^-$, the theory turns out to be the Riordan representation of Pascal matrix in [12].

Acknowledgment

The authors would like to thank the anonymous reviewer for his/her valuable comments and suggestions to improve the quality of the paper.

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